Econometrica, Vol. 49, No. 6 (November, 1981)

ON THE RELATIONSHIPS AMONG SEVERAL SPECIFICATION ERROR TESTS PRESENTED BY DURBIN, WU, AND HAUSMAN[1]

BY ALICE NAKAMURA AND MASAO NAKAMURA

1. INTRODUCTION

IN THE STANDARD LINEAR REGRESSION MODEL it is generally assumed that the regressors are statistically independent of the disturbance term. When this assumption is not appropriate, the use of OLS (ordinary least squares) leads to biased and inconsistent parameter estimates, and the usual t and F tests for these parameters are no longer appropriate. In this case, it may be possible to obtain consistent parameter estimates by the use of an IV (instrumental variables) estimator. However, as Durbin and others have pointed out [**1**, p. 27]: "Since the use of an instrumental variable involves a certain loss of efficiency one should feel rather cautious about using it until the extent of the bias in the ordinary least-squares estimators has been investigated." Thus in his 1954 paper, Durbin [**1**, pp. 28–29] proposed a test for this specification error[2] which involves computing the OLS and IV estimates for the parameter, or parameters, of interest, together with their standard errors, and hence clearly can be carried out using a standard regression package.

Addressing this same problem, Wu [**16**, **17**], in two articles in *Econometrica* in 1973 and 1974, proposed four tests for this specification error which may be applied in any situation where instrumental variables exist for the relevant regressors. He called his test statistics T_1, T_2, T_3, and T_4; and in his articles he presented theoretical and Monte Carlo results indicating that the test based on his statistic T_2 is to be preferred to his other three tests. Wu's statistics have rarely been used in empirical studies in the form in which he presented them, probably because as given by Wu these statistics appear computationally cumbersome. However, a fair amount of theoretical work has been done relating to Wu's T_2. (See, for instance, Farebrother [**3**], Feldstein [**4**], Fuller [**5**], Kariya and Hodoshima [**8**], and Reynolds [**12**].)

In a recent article in *Econometrica*, Hausman [**6**] presents a procedure for testing whether the regressors are independent of the equation disturbance term which can be carried out using any asymptotically efficient estimator and some consistent but asymptotically inefficient estimator.[3] He then gives theoretical results for what he calls an IV version of this test, and presents a closely related test which can be simply carried out using any regression package such as TSP which allows retrieval of results from a previous regression. Hausman shows that the numerator of this second statistic is identical to the numerator of his IV statistic for the special case of a model with one included endogenous variable and containing no exogenous variables. As presented in this portion of his paper, both of Hausman's test statistics are given in forms appropriate for choosing between OLS and IV estimation, which is the context in which Durbin and Wu also presented their statistics.

Neither Wu nor Hausman make any mention of Durbin's 1954 paper in their published work, although they both cite a paper by Liviatan [**10**] which does cite Durbin's 1954

[1] Research supported in part by Canada Council Research Grant 410-77-0339-R1 and by a Canada Council Leave Fellowship for the 1978–79 academic year. We are indebted to Takeshi Amemiya, J. A. Hausman, De-Min Wu, Arnold Zellner, and *Econometrica*'s editors and referees for comments and suggestions on earlier versions of this paper, and to James Heckman, Nicholas Kiefer, and Harvey Rosen for helpful discussions. We are, of course, responsible for any remaining errors or misinterpretations.

[2] We are indebted to James Heckman for bringing this reference to our attention.

[3] The more general context in which Hausman places this testing procedure has led to a number of interesting applications to problems other than the historical choice between OLS and some IV estimator. See, for instance, Hausman [**6**, pp. 1267–1269], Fair and Parke [**2**], and Taylor [**15**].

paper and which uses his test. Also it is clear from certain remarks in the text and footnotes of Hausman's paper that he is aware that there is some relationship between his IV statistic, the computationally more convenient statistic he presents, and one or more of the four statistics presented by Wu. However, the relationships between his own test statistics and those of Wu are not explored or clarified in Hausman's paper.

In this paper we show that the computationally more convenient statistic presented by Hausman is identical to Wu's T_2.[4] As a corollary of this result we are able to give a regression interpretation of $Q_4 - Q^*$, which is the denominator of Wu's T_2. We also note that the IV statistic presented by Hausman is identical to Durbin's test statistic, and that, depending on the estimator used for the nuisance parameter σ^2, both of these test statistics are identical to either Wu's T_3 or T_4 statistic. We conclude the paper with clarifying remarks concerning the χ^2 and F null distributions which Hausman [**6**, p. 1256, Theorem 2.1 and top of p. 1250] and Wu [**16**, p. 737] have derived, respectively, for the statistic T_2, and with some remarks concerning the properties of this statistic and T_4.

2. PROOF OF THE EQUALITY OF HAUSMAN'S STATISTIC AND WU'S T_2

The model is

$$y_1 = Y_2\beta + Z_1\gamma + u \quad \text{(structural equation)}, \tag{1}$$

$$Y_2 = Z_1\Pi_1 + Z_2\Pi_2 + V = Z\Pi + V \quad \text{(reduced form equations)}, \tag{2}$$

where y_1 is $N \times 1$; Y_2 is an $N \times G$ matrix of stochastic regressors; Z_1 and Z_2 are $N \times K_1$ and $N \times K_2$ matrices of instrumental variables; u and V are $N \times 1$ and $N \times G$ matrices of disturbances; and β, γ, Π_1, and Π_2 are $G \times 1$, $K_1 \times 1$, $K_1 \times G$, and $K_2 \times G$ matrices of unknown constants, respectively. Each row of (u, V) is distributed independently with mean zero and the covariance matrix,

$$\Sigma = \begin{pmatrix} \sigma & \delta' \\ \delta & \Sigma_{22} \end{pmatrix}.$$

Also $E(Z_1 u) = E(Z_2 u) = 0$.

Wu's statistic T_2 for testing $H_0 : \delta = 0$ in this model may be written as [**16**, **17**]

$$T_2 = \frac{Q^*}{Q_4 - Q^*}\left(\frac{N - K_1 - 2G}{G}\right), \tag{3}$$

where

$$Q^* = (b_1 - b_2)'\left[(Y_2'A_2Y_2)^{-1} - (Y_2'A_1Y_2)^{-1}\right]^{-1}(b_1 - b_2),$$

$$b_i = (Y_2'A_iY_2)^{-1}Y_2'A_iy_1 \quad \text{for } i = 1, 2,$$

$$A_1 = I - Z_1(Z_1'Z_1)^{-1}Z_1',$$

$$A_2 = Z(Z'Z)^{-1}Z' - Z_1(Z_1'Z_1)^{-1}Z_1',$$

$$Z = (Z_1, Z_2), \quad \text{and } Q_4 = (y_1 - Y_2b_1)'A_1(y_1 - Y_2b_1).$$

[4] A few researchers have used the computationally convenient form of Wu's T_2 presented by Hausman in empirical studies, calling it "the Wu test" [7, **9**, **14**]. However, no proof is given or referenced in these studies showing that this test is, in fact, identical to Wu's T_2 test.

As Wu notes [**16**, pp. 735–736], b_1 is the OLS estimator of β and b_2 is the 2SLS (two-stage least squares) estimator of β for equation (1), while Q_4 is the OLS sum of the squared residuals for (1).

What Hausman refers to as his computationally more convenient test for $H_0: \delta = 0$ is obtained by noting that (1) can be rewritten as

$$y_1 = \hat{Y}_2 \beta_2 + Z_1 \gamma + \hat{e}_2 \beta_3 + u, \tag{4}$$

where $\beta_2 = \beta_3 = \beta$, $Y_2 = \hat{Y}_2 + \hat{e}_2$, $\hat{Y}_2 = Z\hat{\Pi}$, and $\hat{\Pi}$ is the OLS estimator of $\Pi = (\Pi_1, \Pi_2)$ in (2). We will denote the OLS estimators of β_2, γ, and β_3 in (4) by b_2, $\hat{\gamma}$ and b_3, respectively. Since the OLS residuals from equation (2), $\hat{e}_2$, are orthogonal by construction to both $\hat{Y}_2$ and Z_1, the OLS estimators of β_2 and γ in (4) will be identical to the 2SLS estimators of β and γ in (1). Under $H_0: \delta = 0$, $\hat{e}_2$ and u are also statistically independent, and we have $\text{plim}(b_2 - b_3) = \beta_2 - \beta_3 = 0$. Under the alternative hypothesis we still have $\text{plim}\, b_2 = \beta_2$, but $\text{plim}\, b_3 \neq \beta_3 (= \beta_2)$.

An equation which is equivalent to (4) is

$$y_1 = Y_2 \beta_4 + Z_1 \gamma + \hat{e}_2 \beta_5 + u, \tag{5}$$

where $\beta_4 = \beta$ and $\beta_5 = \beta_3 - \beta_2 = 0$. Denoting the OLS estimator of β_5 by b_5, under the null hypothesis $\text{plim}\, b_5 = \beta_5$, where $\beta_5 = 0$ whether the null hypothesis is true or false. The standard statistic for testing this linear restriction may be written as

$$L = \frac{\text{RRSS} - \text{URSS}}{\text{URSS}} \cdot \frac{N - 2G - K_1}{G}, \tag{6}$$

where RRSS, the restricted residual sum of squares, is the residual sum of squares from the regression of y_1 on Y_2 and Z_1. Hence RRSS is equal to Q_4. And URSS, the unrestricted residual sum of squares, is the residual sum of squares from the regression of y_1 on Y_2, Z_1 and $\hat{e}_2$.

Equations (4) and (5) are also equivalent to the following equation used by Hausman [**6**]:

$$y_1 = \hat{Y}_2 \alpha_1 + Y_2 \alpha_2 + Z_1 \gamma + u, \tag{7}$$

where $\alpha_1 = \beta_2 - \beta_3 = 0$ and $\alpha_2 = \beta_3 = \beta$, the OLS estimators of α_1 and α_2 are denoted by $\hat{\alpha}_1$ and $\hat{\alpha}_2$ respectively, and $\text{plim}\, \hat{\alpha}_1 = 0$ under the null hypothesis. It is easily seen that RRSS is the same for equations (4), (5), and (7), since when the appropriate linear restriction is imposed each of these equations reduces to the regression of y_1 on Y_2 and Z_1. To see that URSS is the same for all three equations, we need only note the following relationships between the OLS estimators of the coefficients of these equations: $\hat{\alpha}_1 = b_2 - b_3$, $\hat{\alpha}_2 = b_3$, $b_4 = b_2$, and $b_5 = b_3 - b_2 = -\hat{\alpha}_1$. Also the OLS estimator $\hat{\gamma}$ of γ is identical for all three equations. Hence the OLS residual series will be identical for equations (4), (5), and (7). Since RRSS and URSS are the same for all three equations, the test statistic L given by (6) will also be the same for all three equations.

Using results from Farebrother [**3**, eq. (9)] we have

$$T_2 = \frac{\hat{\epsilon}'\hat{\epsilon} - \hat{\epsilon}_1'\hat{\epsilon}_1 - \hat{\epsilon}_2'\hat{\epsilon}_2}{\hat{\epsilon}_1'\hat{\epsilon}_1 + \hat{\epsilon}_2'\hat{\epsilon}_2} \cdot \frac{N - 2K - K_1}{G} \tag{8}$$

where

$$\hat{\epsilon}'\hat{\epsilon} = Q_4 = \text{RRSS}, \tag{9}$$

$$\hat{\epsilon}_1'\hat{\epsilon}_1 = y_1^{*\prime}(I - X_1(X_1'X_1)^{-1}X_1')y_1^*, \tag{10}$$

$$\hat{\epsilon}_2'\hat{\epsilon}_2 = y_2^{*\prime}(I - X_2(X_2'X_2)^{-1}X_2')y_2^*, \tag{11}$$

and where y_1^* and X_1, and y_2^* and X_2, are alternative transformations of y_1 and Y_2 in the original model such that $y_1^* = Z_3' y_1$, $X_1 = Z_3' Y_2$, $y_2^* = S' y_1$, and $X_2 = S' Y$ with $Z_3 Z_3' = A_3 = I - Z(Z'Z)^{-1}Z'$ and $SS' = A_2$. Thus in order to show that the linear restriction test statistic L given by (6) is identical to Wu's T_2, we need only show that URSS $= \hat{\epsilon}_1'\hat{\epsilon}_1 + \hat{\epsilon}_2'\hat{\epsilon}_2$.

Since $\hat{e}_2'\hat{Y}_2 = \hat{e}_2 Z_1 = 0$, we have

$$\text{URSS} = y_1'\big(I - \hat{e}_2(\hat{e}_2'\hat{e}_2)^{-1}\hat{e}_2' - Z_5(Z_5'Z_5)^{-1}Z_5'\big)y_1, \tag{12}$$

where $Z_5 = (\hat{Y}_2, Z_1)$. But we also have[5]

$$\begin{aligned} Z_5(Z_5'Z_5)^{-1}Z_5' &= (\hat{Y}_2, Z_1)\begin{bmatrix} \hat{Y}_2'\hat{Y}_2 & \hat{Y}_2'Z_1 \\ Z_1'\hat{Y}_2 & Z_1'Z_1 \end{bmatrix}^{-1}\begin{pmatrix} \hat{Y}_2' \\ Z_1' \end{pmatrix} \\ &= Z_1(Z_1'Z_1)^{-1}Z_1' + \hat{Y}_2 D\hat{Y}_2' - Z_1(Z_1'Z_1)^{-1}Z_1'Y_2 D\hat{Y}_2' \\ &\quad - \hat{Y}_2 D Y_2' Z_1(Z_1'Z_1)^{-1}Z_1' + Z_1(Z_1'Z_1)^{-1}Z_1'Y_2 D Y_2' Z_1(Z_1'Z_1)^{-1}Z_1' \\ &= Z_1(Z_1'Z_1)^{-1}Z_1' + Z(Z'Z)^{-1}Z'Y_2 D Y_2' Z(Z'Z)^{-1}Z' \\ &\quad - Z_1(Z_1'Z_1)^{-1}Z_1'Y_2 D Y_2' Z(Z'Z)^{-1}Z' \\ &\quad - Z(Z'Z)^{-1}Z'Y_2 D Y_2' Z_1(Z_1'Z_1)^{-1}Z_1' \\ &\quad + Z_1(Z_1'Z_1)^{-1}Z_1'Y_2 D Y_2' Z_1(Z_1'Z_1)^{-1}Z_1' \\ &= Z_1(Z_1'Z_1)^{-1}Z_1' + \big(Z(Z'Z)^{-1}Z' - Z_1(Z_1'Z_1)^{-1}Z_1'\big) \\ &\quad \times Y_2 D Y_2'\big(Z(Z'Z)^{-1}Z' - Z_1(Z_1'Z_1)^{-1}Z_1'\big) \\ &= Z_1(Z_1'Z_1)^{-1}Z_1' + A_2'Y_2 D Y_2' A_2 = Z_1(Z_1'Z_1)^{-1}Z_1' + A_2'Y_2(Y_2'A_2Y_2)^{-1}Y_2'A_2, \end{aligned} \tag{13}$$

where $D = (\hat{Y}_2'\hat{Y}_2 - Y_2'Z_1(Z_1'Z_1)^{-1}Z_1'Y_2)^{-1} = (Y_2'A_2Y_2)^{-1}$. Thus, combining (12) and (13), we get

$$\begin{aligned} \text{URSS} &= y_1'\big(I - \hat{e}_2(\hat{e}_2'\hat{e}_2)^{-1}\hat{e}_2'\big)y_1 - y_1'Z_1(Z_1'Z_1)^{-1}Z_1'y_1 \\ &\quad - y_1'A_2'Y_2(Y_2'A_2Y_2)^{-1}Y_2'A_2 y_1. \end{aligned} \tag{14}$$

On the other hand, we have from (10) that

$$\begin{aligned} \hat{\epsilon}_1'\hat{\epsilon}_1 &= y_1'A_3y_1 - y_1'A_3Y_2(Y_2'A_3Y_2)^{-1}Y_2'A_3y_1 \\ &= y_1'\big(I - \hat{e}_2(\hat{e}_2'\hat{e}_2)^{-1}\hat{e}_2'\big)y_1 - y_1'Z(Z'Z)^{-1}Z'y_1 \end{aligned} \tag{15}$$

[5] Note that

$$\begin{bmatrix} \hat{Y}_2'\hat{Y}_2 & \hat{Y}_2'Z_1 \\ Z_1'\hat{Y}_2 & Z_1'Z_1 \end{bmatrix}^{-1} = \begin{bmatrix} D & -D(Y_2'Z_1)(Z_1'Z_1)^{-1} \\ -(Z_1'Z_1)^{-1}Z_1'Y_2 D & (Z_1Z_1)^{-1} + (Z_1'Z_1)^{-1}Z_1'Y_2 D Y_2' Z_1(Z_1'Z_1)^{-1} \end{bmatrix}$$

where $\hat{Y}_2'Z_1 = Y_2'Z_1$ is used.

since $A_3 Y_2 = (I - Z(Z'Z)^{-1}Z')Y_2 = \hat{e}_2$. We also have from (11) that

$$(16) \qquad \hat{\epsilon}_2'\hat{\epsilon}_2 = y_1'\left(A_2 - A_2'Y_2(Y_2'A_2Y_2)^{-1}Y_2'A_2\right)y_1$$
$$= y_1'\left[Z(Z'Z)^{-1}Z' - Z_1(Z_1'Z_1)^{-1}Z_1' - A_2'Y_2(Y_2'A_2Y_2)^{-1}Y_2'A_2\right]y_1$$
$$= y_1'Z(Z'Z)^{-1}Z'y_1 - y_1'Z_1(Z_1'Z_1)^{-1}Z_1'y_1$$
$$- y_1'A_2'Y_2(Y_2'A_2Y_2)^{-1}Y_2'A_2y_1.$$

Thus the desired result follows from (14), (15), and (16). It is also seen that we have $Q_4 - Q^* = \text{URSS}$, or $Q^* = \text{RRSS} - \text{URSS}$. Wu's Q^* is therefore the amount by which the OLS sum of the squared residuals is reduced when OLS is applied to (4), (5), or (7) instead of (1).

Given what we have shown and the appropriate definitions in the relevant papers, it is obvious by inspection that all four of Wu's statistics, Durbin's test statistic, and the IV statistic presented by Hausman share the same numerator Q^*. When the IV estimator from equation (1) is used for σ^2 in the denominators of the Durbin statistic and Hausman's IV statistic, these can both be seen by inspection to be equal to Wu's T_3. When the OLS estimator from equation (1) is used instead, the Durbin and Hausman IV statistics can be seen by inspection to equal Wu's T_4 which has Q_4 in the denominator.[6]

3. CONCLUDING REMARKS

Wu [**16**] makes the additional assumption in defining his model that each row of (u, V) has a multivariate normal distribution. Using this assumption he is able to show that under the null hypothesis the unconditional distribution of T_2 is F with G and $N - K_1 - 2G$ degrees of freedom. Although Hausman employs a normality assumption also when considering questions of power, he shows that this same statistic asymptotically obeys a χ^2 distribution under the null hypothesis when no normality assumption is made [**6**, p. 1256, Theorem 2.1, and top of p. 1260]. Under the same normality assumption as Wu's, Kariya and Hodoshima [**8**] show that Wu's T_2 test is not an unbiased test when the critical point is greater than the ratio between the degrees of freedom for the denominator and for the numerator, and that the biasedness of the T_2 test follows from the fact that the non-null distribution of T_2 conditional on y_1 and Y_2 is a doubly non-central F distribution. They also show the close relationship[7] that exists between Wu's T_2 and a test proposed by Revankar [**13**]. Given the main equality established in this paper, it is clear that these finite-sample properties of Wu's T_2 test are also properties of Hausman's test based on his computationally convenient statistic.[8] Finally it is of interest to note that when the OLS estimator of σ^2 is used in Durbin's statistic and Hausman's IV statistic, the tests based on these two statistics and Wu's T_4 are "Lagrange multiplier" type tests. The tests based on Wu's T_2 and Hausman's computationally more convenient statistic, on the other hand, are "Wald" type tests, since they use the unrestricted estimator of σ^2. None of these tests,

[6] See Durbin [**1**, p. 27 and p. 29], Hausman [**6**; expression (2.7) on p. 1254, footnote 8 on p. 1257, p. 1258, and footnote 11 on p. 1260], and Wu [**16**, p. 735, expression (2.20) on p. 736, expression (3.16) on p. 740, and expression (3.20) on p. 741].

[7] The numerators of these two statistics are identical up to a constant of proportionality.

[8] Nakamura and Nakamura [**11**] demonstrated other finite sample properties of the Wu-Hausman test using Monte Carlo experiments. It also follows easily from [**8**, Eqs. (3.16)–(3.18)] that when (1) is just-identified, Wu's T_2 (and hence Hausman's second statistic), Revankar's, and Revankar and Hartlay's test statistics become identical, for when (1) is just-identified, $K_2 = G_2$ and hence $q_2 = 0$ and $c_1 = c_2 = c_3$ in [**8**] using their notation. See also [**13**, Remark 2, p. 171].

however, are true Lagrange multiplier or Wald tests, since they are not based on maximum likelihood estimates.

University of Alberta

Manuscript received April, 1979; revision received October, 1980.

REFERENCES

[1] DURBIN, J.: "Errors in Variables," *Review of the International Statistical Institute*, 22(1954), 23–32.

[2] FAIR, R. C., AND W. R. PARKE: "Full-Information Estimates of a Nonlinear Macroeconometric Model," *Journal of Econometrics*, 13(1980), 269–291.

[3] FAREBROTHER, R. W.: "A Remark on the Wu Test," *Econometrica*, 44(1976), 475–477.

[4] FELDSTEIN, M.: "Errors in Variables: A Consistent Estimator With Smaller MSE in Finite Samples," *Journal of the American Statistical Association*, 69(1974), 990–996.

[5] FULLER, W. A.: "Some Properties of a Modification of the Limited Information Estimator," *Econometrica*, 45(1977), 939–953.

[6] HAUSMAN, J. A.: "Specification Tests in Econometrics," *Econometrica*, 46(1978), 1251–1271.

[7] HECKMAN, J. J.: "Sample Selection Bias as a Specification Error with an Application to the Estimation of Labor Supply Functions," Working Paper No. 172, National Bureau of Economic Research, March, 1977, published in *Female Labor Supply: Theory and Estimation*, ed. by J. P. Smith. Princeton: Princeton University Press, 1980.

[8] KARIYA, T., AND J. HODOSHIMA: "Finite Sample Properties of the Tests for Independence in Structural Systems and the LRT," *Economic Studies Quarterly*, 31(1980), 45–56.

[9] KIEFER, N. M.: "Population Heterogeneity and Inference from Panel Data on the Effects of Vocational Education," *Journal of Political Economy*, 87(1979), S213–S226.

[10] LIVIATAN, N.: "Tests of the Permanent Income Hypotheses Based on a Re-Interview Savings Study," in *Measurement in Economics*, ed. by C. Christ. Stanford: Stanford University Press, 1963, pp. 29–59.

[11] NAKAMURA, A., AND M. NAKAMURA: "On the Usefulness of the Wu-Hausman Test for Detecting the Least Squares Bias Problem," paper presented at the World Congress of the Econometric Society, Aix-en-Provence, August, 1980.

[12] REYNOLDS, R.: "Posterior Odds for the Hypothesis of Independence Between Stochastic Regressors and Disturbances," Graduate School of Business, University of Chicago, 1979.

[13] REVANKAR, N. S.: "Asymptotic Relative Efficiency Analysis of Certain Tests of Independence in Structural Systems," *International Economic Review*, 19(1978), 165–179.

[14] ROSEN, H. S., AND K. T. ROSEN: "Federal Taxes and Homeownership: Evidence from Time Series," *Journal of Political Economy*, 88(1980), 59–75.

[15] TAYLOR, W. E.: "Small Sample Considerations in Estimation from Panel Data," *Journal of Econometrics*, 13(1980), 203–223.

[16] WU, D.: "Alternative Tests of Independence Between Stochastic Regressors and Disturbances," *Econometrica*, 41(1973), 733–750.

[17] ———: "Alternative Tests of Independence Between Stochastic Regressors and Disturbances: Finite Sample Results," *Econometrica*, 42(1974), 529–546.